

The Critical Exponent of Doubly Singular Parabolic Equations¹

Xinfeng Liu and Mingxin Wang

*Department of Applied Mathematics, Southeast University,
Nanjing 210018, People's Republic of China
E-mail: mxwang@seu.edu.cn*

Submitted by Howard Levine

Received June 24, 1999

In this paper we study the Cauchy problem of doubly singular parabolic equations $u_t = \operatorname{div}(|\nabla u|^\sigma \nabla u^m) + t^s |x|^\theta u^p$ with non-negative initial data. Here $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma + 2)/N\}$ satisfying $0 < \sigma + m \leq 1$, $p > 1$, and $s \geq 0$. We prove that if $\theta > \max\{-(\sigma + 2), (1 + s)[N(1 - \sigma - m) - (\sigma + 2)]\}$, then $p_c = (\sigma + m) + (\sigma + m - 1)s + [(\sigma + 2)(1 + s) + \theta]/N > 1$ is the critical exponent; i.e., if $1 < p \leq p_c$ then every non-trivial solution blows up in finite time. But for $p > p_c$ a positive global solution exists. © 2001 Academic Press

Key Words: doubly singular parabolic equation; critical exponent; blow up.

1. INTRODUCTION

In this paper we study critical exponent of quasilinear parabolic equations

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^\sigma \nabla u^m) + t^s |x|^\theta u^p, & x \in \mathbb{R}^N, & \quad t > 0, \\ u(x, 0) &= u_0(x) \geq 0, & x \in \mathbb{R}^N, & \end{aligned} \quad (1.1)$$

where $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma + 2)/N\}$ satisfying $0 < \sigma + m \leq 1$, $p > 1$, and $s \geq 0$. $u_0(x)$ is a continuous function in \mathbb{R}^N . The existence, uniqueness, and comparison principle for the solution to (1.1) had been proved in [11] (for the definition of solution see [11]). Since $0 < \sigma + m \leq 1$, (1.1) is a doubly singular problem and does not have finite speed of propagation. Therefore, $u(x, t) > 0$ for all $x \in \mathbb{R}^N$ and $t > 0$.

¹This project was supported by PRC Grant NSFC 19831060.



Because the main interests of this paper are to study the large-time behavior of solution, we assume that the solution u of (1.1) has very mild regularity. In this context, “ $u(x, t)$ blows up in finite time” means that $w(t) = \int_{\Omega} u(x, t) dx \rightarrow +\infty$ as $t \rightarrow T^-$ for some finite time $T > 0$, where Ω is a bounded domain in R^N .

Our main result reads as follows:

THEOREM 1. *Assume that $s \geq 0$, $p > 1$, $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma + 2)/N\}$ satisfying $0 < \sigma + m \leq 1$. If $\theta > \max\{-(\sigma + 2), (1 + s)[N(1 - \sigma - m) - (\sigma + 2)]\}$, then $p_c = (\sigma + m) + (\sigma + m - 1)s + [(\sigma + 2)(1 + s) + \theta]/N > 1$ is the critical exponent; i.e, if $1 < p \leq p_c$ then every non-trivial solution of (1.1) blows up in finite time, whereas if $p > p_c$ then (1.1) has a small non-trivial global solution.*

The study of blow-up for nonlinear parabolic equations probably originates from Fujita [8], where he studied the Cauchy problem of the semilinear heat equation,

$$\begin{aligned} u_t &= \Delta u + u^p, & x &\in R^N, & t &> 0, \\ u(x, 0) &= u_0(x) \geq 0, & x &\in R^N, \end{aligned} \quad (1.2)$$

where $p > 1$, and obtained the following results:

(a) If $1 < p < 1 + 2/N$, then every nontrivial solution $u(x, t)$ blows up in finite time.

(b) If $p > 1 + 2/N$ and $u_0(x) \leq \delta e^{-|x|^2}$ ($0 < \delta \ll 1$), then (1.2) admits a global solution.

In the critical case $p = 1 + 2/N$, it was shown by Hayakawa [10] for dimensions $N=1, 2$ and by Kobayashi *et al.* [12] for all $N \geq 1$ that (1.2) possesses no global solution $u(x, t)$ satisfying $\|u(\cdot, t)\|_{\infty} < \infty$ for $t \geq 0$. Weissler [24] proved that if $p = 1 + 2/N$, then (1.2) possesses no global solution $u(x, t)$ satisfying $\|u(\cdot, t)\|_q < \infty$ for $t > 0$ and some $q \in [1, +\infty)$. The value $p_c = 1 + 2/N$ is called the critical exponent of (1.2). It plays an important role in studying the behavior of the solution to (1.2).

In the past couple of years there have been a number of extensions of Fujita's results in several directions. These include similar results for other geometries (cones and exterior domains) [4, 5, 13, 15, 16], quasilinear parabolic equations, and systems [1, 2, 5, 7, 9, 14, 18–20, 22, 23]. In particular, the authors of [2] considered degenerate equations on domains with non-compact boundary. There are also results for nonlinear wave equations and nonlinear Schrödinger equations. We refer the reader to the survey papers by Deng and Levine [5] and Levine [13] for a detailed account of this aspect.

When $m = 1$, (1.1) becomes p -Laplacian equations, and the critical exponents were given by the authors of [19, 21, 22]. When $\sigma = 0$, (1.1) becomes the porous media equations, and the critical exponents were studied by the authors of [13, 17, 18, 22].

This paper is organized as follows. In Section 2 we discuss the qualitative behaviors and give some estimates of solutions to the homogeneous problem

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^\sigma \nabla u^m), & x &\in R^N, & t &> 0, \\ u(x, 0) &= u_0(x) \geq, \neq 0, & x &\in R^N. \end{aligned} \quad (1.3)$$

In Section 3, for convenience, we first discuss the special case of (1.1): $s = 0$, i.e.,

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^\sigma \nabla u^m) + |x|^\theta u^p, & x &\in R^N, & t &> 0, \\ u(x, 0) &= u_0(x) \geq, \neq 0, & x &\in R^N, \end{aligned} \quad (1.4)$$

and prove that if $1 < p \leq \tilde{p}_c \stackrel{\Delta}{=} \sigma + m + (\sigma + 2 + \theta)/N$ then every non-trivial solution of (1.4) blows up in finite time. In Section 4 we prove Theorem 1.

Remark. We end this section with a simple but very useful reduction. When we consider the blow-up case, by the comparison principle we need only consider that $u_0(x)$ is radially symmetric and non-increasing, i.e., $u_0(x) = u_0(r)$ with $r = |x|$, and $u_0(r)$ is non-increasing in r . Therefore, the solution of (1.1) is also radially symmetric and non-increasing in $r = |x|$.

2. ESTIMATES OF SOLUTIONS TO (1.3)

In this section we discuss (1.3) for the radially symmetric case; the main results are three propositions.

PROPOSITION 1. *Assume that $-1 < \sigma \leq 0$ and $m > 1 - \sigma - (\sigma + 2)/N$ satisfy $0 < \sigma + m \leq 1$.*

(i) *If $\sigma + m < 1$, then, for any $c > 0$, the equation (1.3) has a global self-similar solution,*

$$u(x, t) = ct^{-\alpha}(1 + hr^\nu)^{-q},$$

where $\alpha = N/[N(\sigma + m - 1) + \sigma + 2]$, $\beta = 1/[N(\sigma + m - 1) + \sigma + 2]$, $\nu = (\sigma + 2)/(\sigma + 1)$, $q = (\sigma + 1)/(1 - \sigma - m)$, $r = |x|t^{-\beta}$, and $h = h(c) = \frac{1}{qv} \beta^{1/(1+\alpha)} c^{(1-\sigma-m)/(\sigma+1)} m^{-1/(\sigma+1)}$.

(ii) If $\sigma + m = 1$, then, for any $c > 0$, the equation (1.3) has a global self-similar solution,

$$u(x, t) = ct^{-\alpha} \exp\{-hr^\nu\},$$

where $\alpha = N/(\sigma + 2)$, $\beta = 1/(\sigma + 2)$, $\nu = (\sigma + 2)/(\sigma + 1)$, $r = |x|t^{-\beta}$, and h satisfies $(h\nu)^{\sigma+1} = \beta/m$.

This proposition can be verified directly.

PROPOSITION 2. Assume that $-1 < \sigma \leq 0$, $m > \max\{0, 1 - \sigma - (\sigma + 2)/N\}$, such that $0 < \sigma + m \leq 1$ and $u_0(x)$ is a non-trivial and non-negative continuous function. If $u_0(x)$ is a radially symmetric and non-increasing function, then the solution $u(x, t)$ of (1.3) satisfies

$$u_t \geq -\frac{\alpha}{t}u \quad \text{for all } x \in R^N, \quad t > 0, \quad (2.1)$$

where $\alpha = N/[N(\sigma + m - 1) + \sigma + 2]$.

Proof. Denote $k = (\sigma + m)/(\sigma + 1)$, let $f = (mk^{-(\sigma+1)})^{1/(k\sigma+k-1)}u$ when $\sigma + m < 1$, and let $f = u$ when $\sigma + m = 1$. Then (1.3) can be rewritten as

$$\begin{aligned} f_t &= d \operatorname{div}(|\nabla f^k|^\sigma \nabla f^k), & x \in R^N, & \quad t > 0, \\ f(x, 0) &= f_0(x) \geq, \neq 0, & x \in R^N, \end{aligned}$$

where $d = 1$ when $\sigma + m < 1$ and $d = mk^{-(\sigma+1)}$ when $\sigma + m = 1$. Let $g = f^k$; then g satisfies the following equation:

$$\begin{aligned} g_t^{1/k} &= d \operatorname{div}(|\nabla g|^\sigma \nabla g), & x \in R^N, & \quad t > 0, \\ g(x, 0) &= f_0^k(x) \geq, \neq 0, & x \in R^N. \end{aligned}$$

Denote $\mu = (1 + \sigma - 1/k)/(\sigma + 1)$ if $\sigma + m < 1$, and let

$$v = \begin{cases} \frac{1}{\mu} g^\mu & \text{if } 0 < \sigma + m < 1, \\ \ln g & \text{if } \sigma + m = 1. \end{cases}$$

Case 1. $0 < \sigma + m < 1$. In this case, $d = 1$ and g satisfies

$$\begin{aligned} g_t &= kg \operatorname{div}(|\nabla v|^\sigma \nabla v) + g^{-1/k} |\nabla g|^{\sigma+2} \geq kg \operatorname{div}(|\nabla v|^\sigma \nabla v), \\ v_t &= g^{-1/k(\sigma+1)} g_t = kg^{-1/k(\sigma+1)} g^{1-1/k} \operatorname{div}(|\nabla g|^\sigma \nabla g) \\ &= k\mu v \operatorname{div}(|\nabla v|^\sigma \nabla v) + |\nabla v|^{\sigma+2}. \end{aligned} \quad (2.2)$$

Denote $w = \operatorname{div}(|\nabla v|^\sigma \nabla v)$, $\partial/\partial r = '$ and let $z = -v$; then $z > 0$, $z' > 0$, and

$$\begin{aligned}
 z_t &= -k\mu z \operatorname{div}(|\nabla z|^\sigma \nabla z) - |\nabla z|^{\sigma+2} \\
 &= -k\mu z \left[(\sigma+1)(z')^\sigma z'' + \frac{N-1}{r}(z')^{\sigma+1} \right] - (z')^{\sigma+2} = k\mu zw - (z')^{\sigma+2}, \\
 w &= - \left[(\sigma+1)(z')^\sigma z'' + \frac{N-1}{r}(z')^{\sigma+1} \right], \\
 -w_t &= (\sigma+1)(z')^\sigma z'_t + (\sigma+1)\sigma(z')^{\sigma-1} z'_t z'' + \frac{(N-1)(\sigma+1)}{r}(z')^\sigma z'_t, \quad (2.3) \\
 z'_t &= k\mu(z'w + zw') - (\sigma+2)(z')^{\sigma+1} z'', \\
 z''_t &= k\mu(wz'' + 2w'z' + w''z) - (\sigma+2)[(z')^{\sigma+1} z''' + (\sigma+1)(z')^\sigma (z'')^2].
 \end{aligned}$$

By a series of calculation we have

$$\begin{aligned}
 -w_t &= k\mu(\sigma+1) \left[z(z')^\sigma \Delta w + 2(z')^{\sigma+1} w' + (\sigma+1)(z')^\sigma w z'' \right. \\
 &\quad \left. + \sigma(z')^{\sigma-1} z z'' w' + \frac{N-1}{r}(z')^{\sigma+1} w + \frac{N-1}{r}(z')^\sigma z w' \right] \\
 &\quad - (\sigma+1)(\sigma+2) \left[(z')^{2\sigma+1} z''' + (1+2\sigma)(z')^{2\sigma} (z'')^2 \right. \\
 &\quad \left. + \frac{N-1}{r}(z')^{2\sigma+1} z'' \right]. \quad (2.4)
 \end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}
 -w' &= \sigma(\sigma+1)(z')^{\sigma-1} (z'')^2 + (\sigma+1)(z')^\sigma z''' \\
 &\quad - \frac{N-1}{r^2} (z')^{\sigma+1} + \frac{(N-1)(\sigma+1)}{r} (z')^\sigma z''.
 \end{aligned}$$

Denote $\varepsilon = k\mu(\sigma+1) = k(1+\sigma-1/k)$; substituting the above expression into (2.4) we get

$$\begin{aligned}
 -w_t &= \varepsilon a(r, t) \Delta w + b(r, t) w' - \varepsilon w^2 - (\sigma+2) \\
 &\quad \times \left[\frac{N-1}{r^2} (z')^{2\sigma+2} - (\sigma+1)(z')^{2\sigma} (z'')^2 \right] \\
 &= \varepsilon a(r, t) \Delta w + b(r, t) w' - \varepsilon w^2 + (\sigma+2) \\
 &\quad \times \left[(\sigma+1)w(z')^\sigma z'' - \frac{N-1}{r^2} (z')^{2\sigma+2} + (\sigma+1)\frac{N-1}{r} (z')^{2\sigma+1} z'' \right] \\
 &= \varepsilon a(r, t) \Delta w + b(r, t) w' - \varepsilon w^2 - (\sigma+2) \\
 &\quad \times \left[w^2 + \frac{2(N-1)}{r} (z')^{\sigma+1} w + \frac{N(N-1)}{r^2} (z')^{2\sigma+2} \right],
 \end{aligned}$$

where $a(r, t), b(r, t)$ are functions produced by $z(r, t)$ and $z'(r, t)$. Taking into account the Cauchy inequality

$$-2\frac{N-1}{r}(z')^{\sigma+1}w \leq \frac{N-1}{N}w^2 + \frac{N(N-1)}{r^2}(z')^{2\sigma+2},$$

we have

$$\begin{aligned} -w_t &\leq k(\sigma+1-1/k)a(r, t)\Delta w + b(r, t)w' \\ &\quad + \left[1 - k(\sigma+1) - \frac{\sigma+2}{N}\right]w^2; \end{aligned}$$

i.e.,

$$\begin{aligned} w_t &\geq k(-\sigma-1+1/k)a(r, t)\Delta w - b(r, t)w' \\ &\quad + \left[\frac{\sigma+2}{N} - k(1/k - (\sigma+1))\right]w^2. \end{aligned}$$

Noticing $k = (\sigma+m)/(\sigma+1)$, we have

$$\begin{aligned} w_t &\geq k[1/k - (\sigma+1)]a(r, t)\Delta w - b(r, t)w' \\ &\quad + \frac{\sigma+2+N(\sigma+m-1)}{N}w^2. \end{aligned}$$

Let $y(r, t) = -\alpha/t$. It is obvious that $y_t = k[1/k - (\sigma+1)]a(r, t)\Delta y - b(r, t)y' + y^2/\alpha$. Since $y(r, 0) = -\infty$, it follows by the comparison principle that $w \geq -\alpha/t$ (see [3, 11]); i.e., $\operatorname{div}(|\nabla v|^\sigma \nabla v) \geq -\alpha/t$. By (2.2) we have $g_t \geq -k\alpha g/t$. Since $g = f^k$, it follows that $f_t = -\alpha f/t$; i.e.

$$u_t \geq -\frac{\alpha}{t}u.$$

Case 2. $\sigma+m=1$. Since this is easy to prove, we omit the details here. Q.E.D.

Remark. For the porous media equation, the authors of [3] proved (2.1) for first time, to our knowledge.

PROPOSITION 3. *Under the assumptions of Propositions 1 and 2, there exist positive constants δ, b such that:*

(i) *When $\sigma+m < 1$, then*

$$u(x, t) \geq \delta(t-\varepsilon)^{-\alpha}(1+br^\nu)^{-q} \quad \forall |x| > 1, \quad t > \varepsilon > 0, \quad (2.5)$$

where $r = |x|(t-\varepsilon)^{-\beta}$, α, β, ν , and q are as in Proposition 1, and b is a positive constant.

(ii) When $\sigma + m = 1$, then

$$u(x, t) \geq \delta(t - \varepsilon)^{-\alpha} \exp\{-br^\nu\} \quad \forall |x| > 1, \quad t > \varepsilon > 0, \quad (2.6)$$

where $r = |x|(t - \varepsilon)^{-\beta}$, α , β , and ν are as in Proposition 1, and b is a positive constant.

Proof. In view of Propositions 1 and 2, and using a method similar to that of [21], one can prove Proposition 3. Here we give only the sketch of the proof for the case $\sigma + m < 1$.

Step 1. By use of the methods of Chap. 6 of [6] we can prove the following comparison lemma:

LEMMA 1. Let $0 \leq \tau < +\infty$ and $S = \{x \in R^N, |x| > 1\} \times [\tau, +\infty)$. Assume that v, w are non-negative functions satisfying

$$\begin{aligned} v_t &= \operatorname{div}(|\nabla v|^\sigma \nabla v^m), & w_t &= \operatorname{div}(|\nabla w|^\sigma \nabla w^m) && \text{in } S, \\ v(x, t) &\leq w(x, t), & |x| &= 1, && \tau < t < +\infty, \\ v(x, \tau) &\leq w(x, \tau), & |x| &\geq 1. \end{aligned}$$

Then

$$v(x, t) \leq w(x, t) \quad \text{in } S.$$

Step 2. From Proposition 1 we have that problem (1.3) has the similarity solutions

$$U_\mu(x, t) = \mu^\rho U(\mu x, t), \quad \rho = (\sigma + 2)/(1 - \sigma - m),$$

where $\mu > 0$ is a parameter, and

$$U(x, t) = U_1(x, t) = t^{-\alpha}(1 + hr^\nu)^{-q}, \quad r = |x|t^{-\beta}.$$

In view of Proposition 2 and the expression of $U_\mu(x, t)$ we can prove that for suitably small $\mu > 0$, the following holds:

$$\begin{aligned} U_\mu(1, t - \varepsilon) &\leq u(1, t) && \text{for } t > \varepsilon, \\ U_\mu(x, t - \varepsilon) &= 0 \leq u(x, t) && \text{for } |x| \geq 1, \quad t = \varepsilon. \end{aligned}$$

By Lemma 1 we see that (2.5) holds.

Q.E.D.

3. THE SPECIAL CASE $s = 0$, $1 < p \leq \tilde{p}_c$

In this section we study problem (1.4) and prove a blow-up result.

THEOREM 2. *Let σ, m, p, θ be as in Theorem 1. If $1 < p \leq \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$, then every non-trivial solution of (1.4) blows up in finite time.*

Let $\phi(x)$ be a smooth, radially symmetric, and non-increasing function which satisfies $0 \leq \phi(x) \leq 1$, $\phi(x) \equiv 1$ for $|x| \leq 1$, and $\phi(x) \equiv 0$ for $|x| \geq 2$. It follows that for $l > 1$, $\phi_l(x) = \phi(x/l)$ is a smooth, radially symmetric, and non-increasing function which satisfies $0 \leq \phi_l(x) \leq 1$, $\phi_l(x) \equiv 1$ for $|x| \leq l$ and $\phi_l(x) \equiv 0$ for $|x| \geq 2l$. It is easy to see that $|\nabla \phi_l| \leq C/l$, $|\Delta \phi_l| \leq C/l^2$. Let

$$w_l(t) = \int_{\Omega} u \phi_l dx,$$

where $\Omega = R^N \setminus B_1$, with B_1 being the unit ball with center at the origin. We divide the argument into two cases.

Case 1. $m \leq 1$. Let $q = (m + \sigma)/(\sigma + 1)$ and $v = u^q$; then the equation (1.4) can be written as

$$(v^{1/q})_t = \frac{m}{q^{\sigma+1}} \operatorname{div}(|\nabla v|^{\sigma} \nabla v) + |x|^{\theta} v^{p/q}.$$

Therefore,

$$\begin{aligned} \frac{dw_l}{dt} &= \frac{m}{q^{\sigma+1}} \int_{\Omega} \operatorname{div}(|\nabla v|^{\sigma} \nabla v) \phi_l dx + \int_{\Omega} |x|^{\theta} v^{p/q} \phi_l dx \\ &\geq -\frac{m}{q^{\sigma+1}} \omega_N \int_1^{2l} |v'|^{\sigma+1} |\phi'_l| r^{N-1} dr + \int_{\Omega} |x|^{\theta} v^{p/q} \phi_l dx. \end{aligned}$$

By direct computation we have

$$\begin{aligned} \int_1^{2l} |v'|^{\sigma+1} |\phi'_l| r^{N-1} dr &\leq \left(\int_1^{2l} |v'| |\phi'_l| r^{N-1} dr \right)^{\sigma+1} \left(\int_1^{2l} r^{N-1} |\phi'_l| dr \right)^{-\sigma}, \\ \int_1^{2l} |v'| |\phi'_l| r^{N-1} dr &= \frac{1}{\omega_n} \int_{\Omega} \nabla v \cdot \nabla \phi_l dx \leq \frac{1}{\omega_N} \int_{\Omega} v |\Delta \phi_l| dx, \\ \int_{\Omega} v |\Delta \phi_l| dx &\leq \left(\int_{\Omega} |x|^{\theta} v^{p/q} \phi_l dx \right)^{q/p} \\ &\quad \times \left(\int_{\Omega} \{ |\Delta \phi_l|^p \phi_l^{-q} |x|^{-\theta q} \}^{1/(p-q)} dx \right)^{(p-q)/p}, \\ \left(\int_{\Omega} \{ |\Delta \phi_l|^p \phi_l^{-q} |x|^{-\theta q} \}^{1/(p-q)} dx \right)^{(p-q)/p} &= C_1 l^{N(p-q)-\theta q-2p/p}, \\ \left(\int_1^{2l} r^{N-1} \phi'_l dr \right)^{-\sigma} &= C_2 l^{-(N-1)\sigma}. \end{aligned}$$

In view of $m \leq 1$, we have $q \leq 1$, and hence $p/q > 1$.

Case 2. $m > 1$. In this case one has

$$\begin{aligned}
 \frac{dw_l}{dt} &= \int_{\Omega} \operatorname{div}(|\nabla u|^{\sigma} \nabla u^m) \phi_l dx + \int_{\Omega} |x|^{\theta} u^p \phi_l dx \\
 &= \int_{\partial\Omega} |\nabla u|^{\sigma} \frac{\partial u^m}{\partial \eta} \phi_l ds - \int_{\Omega} |\nabla u|^{\sigma} \nabla u^m \cdot \nabla \phi_l dx + \int_{\Omega} |x|^{\theta} u^p \phi_l dx \\
 &\geq - \int_{\Omega} |\nabla u|^{\sigma} \nabla u^m \cdot \nabla \phi_l dx + \int_{\Omega} |x|^{\theta} u^p \phi_l dx \\
 &\geq -m\omega_N \int_1^{2l} |u'|^{\sigma+1} u^{m-1} |\phi'_l| r^{N-1} dr + \int_{\Omega} |x|^{\theta} u^p \phi_l dx.
 \end{aligned}$$

By direct computation and using Hölder's inequality one has

$$\begin{aligned}
 \int_1^{2l} |u'|^{\sigma+1} u^{m-1} |\phi'_l| r^{N-1} dr &\leq \left(\int_1^{2l} |u'| r^{N-1} |\phi'_l| dr \right)^{\sigma+1} \\
 &\quad \times \left(\int_1^{2l} |\phi'_l| u^{-(m-1)/\sigma} r^{N-1} dr \right)^{-\sigma}, \\
 \int_1^{2l} |\phi'_l| u^{-(m-1)/\sigma} r^{N-1} dr &= \int_{\Omega} |\nabla \phi_l| u^{-(m-1)/\sigma} dx \\
 &\leq \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{-(m-1)/p\sigma} \left(\int_{\Omega} \{ |x|^{\theta(m-1)} \right. \\
 &\quad \times |\nabla \phi_l|^{p\sigma} \phi_l^{m-1} \}^{1/(m-1+p\sigma)} dx \Big)^{(m-1+p\sigma)/p\sigma}, \\
 \int_1^{2l} |u'| r^{N-1} |\phi'_l| dr &= -\frac{1}{\omega_N} \int_{\Omega} u |\Delta \phi_l| dx \leq \frac{1}{\omega_N} \int_{\Omega} u |\Delta \phi_l| dx, \\
 \int_{\Omega} u |\Delta \phi_l| dx &\leq \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{1/p} \\
 &\quad \times \left(\int_{\Omega} \{ |x|^{-\theta} |\Delta \phi_l|^p \phi_l^{-1} \}^{1/(p-1)} dx \right)^{(p-1)/p}, \\
 \left(\int_{\Omega} \{ |x|^{\theta(m-1)} |\nabla \phi_l|^{p\sigma} \phi_l^{m-1} \}^{1/(m-1+p\sigma)} dx \right)^{(m-1+p\sigma)/p\sigma} \\
 &= C'_1 l^{[\theta(m-1)+N(m-1+p\sigma)-p\sigma]/p\sigma}, \\
 \left(\int_{\Omega} \{ |x|^{-\theta} |\Delta \phi_l|^p \phi_l^{-1} \}^{1/(p-1)} dx \right)^{(p-1)/p} &= C'_2 l^{[N(p-1)-2p-\theta]/p}.
 \end{aligned}$$

In view of $m > 1$, $0 < m + \sigma \leq 1$, it follows that $0 < -(m-1)/\sigma \leq 1$.

For the above two cases we always have

$$\begin{aligned}
 \frac{dw_l}{dt} &\geq -C_3 \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(\sigma+m)/p} l^{-\theta(m+\sigma)/p-2-\sigma+N-N(\sigma+m)/p} \\
 &\quad + \int_{\Omega} |x|^{\theta} u^p \phi_l dx;
 \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{dw_l}{dt} \geq & \left\{ -C_3 l^{-\theta(\sigma+m)/p-2-\sigma+N-N(\sigma+m)/p} + \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(p-\sigma-m)/p} \right\} \\ & \times \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(\sigma+m)/p}. \end{aligned} \quad (3.1)$$

By Hölder's inequality we have

$$\int_{\Omega} |x|^{\theta} u^p \phi_l dx \geq \left(\int_{\Omega} u \phi_l dx \right)^p \left(\int_{\Omega} |x|^{-\theta/(p-1)} \phi_l dx \right)^{-(p-1)}.$$

Hence

$$\int_{\Omega} |x|^{\theta} u^p \phi_l dx \geq \begin{cases} c w_l^p l^{\theta-N(p-1)} & \text{if } \theta < N(p-1), \\ c w_l^p (\ln l)^{-(p-1)} & \text{if } \theta = N(p-1), \\ c w_l^p & \text{if } \theta > N(p-1). \end{cases} \quad (3.2)$$

We now prove Theorem 2.

(i) First we consider the case $\theta < N(p-1)$. It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{dw_l}{dt} \geq & \left\{ -C_3 l^{-\theta(\sigma+m)/p-2-\sigma+N-N(\sigma+m)/p} \right. \\ & \left. + C_4 w_l^{p-(\sigma+m)} l^{[\theta-N(p-1)](p-(\sigma+m))/p} \right\} \\ & \times \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(\sigma+m)/p}. \end{aligned} \quad (3.3)$$

(a) $p < \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$. Under this assumption, one has

$$\{\theta - N(p-1)\} \{p - (\sigma + m)\} / p > N - 2 - \sigma - \{N(\sigma + m) + \theta(m + \sigma)\} / p,$$

and consequently

$$\begin{aligned} l^{\{\theta-N(p-1)\}\{p-(\sigma+m)\}/p} / l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} & \rightarrow +\infty \\ \text{as } l & \rightarrow +\infty. \end{aligned} \quad (3.4)$$

Using the fact that w_l is an increasing function of l , we find from (3.3) and (3.4) that there exist $\delta > 0$, $l \gg 1$ such that

$$\frac{dw_l}{dt} \geq \delta \int_{\Omega} |x|^{\theta} u^p \phi_l dx \geq \delta w_l^p(t) l^{\theta-N(p-1)} \quad \forall t > 0.$$

Thus w_l , and consequently u , blows up in finite time, since $p > 1$.

(b) $p = \tilde{p}_c = \sigma + m + (\sigma + 2 + \theta)/N$. In this case, $\{\theta - N(p - 1)\}\{p - (\sigma + m)\}/p = N - 2 - \sigma - \{N(\sigma + m) + \theta(m + \sigma)\}/p < 0$. If we can prove that

$$\int_{\Omega} u \phi_l dx$$

is a unbounded function of t for some l , then it can be shown that, as in the above case, w_l , and hence u , blows up in finite time. Otherwise, $u(\cdot, t) \in L^1(\Omega)$ for all $t > 0$ and there exists an $M > 0$ such that

$$\|u(t)\|_{L^1(\Omega)} \leq M \quad \text{for all } t > 0. \quad (3.5)$$

We will prove (3.5) is impossible. Suppose the contrary; it is clear from (3.1) that, for the large l , if $\int_{\Omega} |x|^{\theta} u^p dx < +\infty$ then $dw_l/dt \geq \frac{1}{2} \int_{\Omega} |x|^{\theta} u^p \phi_l dx$, and if $\int_{\Omega} |x|^{\theta} u^p dx = +\infty$ then $w'_l(t) \geq 1$. Therefore,

$$w'_l(t) \geq k_l(t) \triangleq \min \left\{ 1, \frac{1}{2} \int_{\Omega} |x|^{\theta} u^p \phi_l dx \right\}, \quad l \gg 1,$$

$$w_l(t) - w_l(0) \geq \int_0^t k_l(\tau) d\tau.$$

Let $w(t) = \int_{\Omega} u(x, t) dx$ and take $l \rightarrow +\infty$ in the above inequality. We obtain

$$w(t) - w(0) \geq \int_0^t k(\tau) d\tau, \quad (3.6)$$

where $k(t) = \min\{1, \frac{1}{2} \int_{R^N} |x|^{\theta} u^p dx\}$. When $\sigma + m < 1$, using (2.5) and by direct computation we have

$$\begin{aligned} \int_{\Omega} |x|^{\theta} u^p dx &\geq \delta^p (t - \varepsilon)^{-1} \int_{|y| \geq (t - \varepsilon)^{-\beta}} |y|^{\theta} (1 + b|y|^{\nu})^{-qp} dy \\ &\geq c(t - \varepsilon)^{-1}, \quad t \gg 1. \end{aligned}$$

When $\sigma + m = 1$, using (2.6) and by direct computation we have

$$\begin{aligned} \int_{\Omega} |x|^{\theta} u^p dx &\geq \delta^p (t - \varepsilon)^{-1} \int_{|y| \geq (t - \varepsilon)^{-\beta}} |y|^{\theta} \exp\{-b|y|^{\nu}\} dy \\ &\geq c(t - \varepsilon)^{-1}, \quad t \gg 1. \end{aligned}$$

In view of (3.6) it yields

$$\lim_{t \rightarrow +\infty} w(t) = +\infty;$$

i.e.,

$$\lim_{t \rightarrow +\infty} \int_{\Omega} u(x, t) dx = +\infty.$$

This shows that (3.5) is impossible. And hence $u(x, t)$ blows up in finite time.

(ii) Next we consider the case $\theta \geq N(p-1)$. Since $m > 1 - \sigma - (\sigma+2)/N$, it follows that $N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p < 0$. Combining (3.2) and (3.1) we find that, for the case $\theta = N(p-1)$,

$$\begin{aligned} \frac{dw_l}{dt} &\geq \left(-C_3 l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} + C w_l^{p-(\sigma+m)} (\ln l)^{\frac{(\sigma+m-p)(p-1)}{p}} \right) \\ &\quad \times \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(\sigma+m)/p}, \end{aligned}$$

and for the case $\theta > N(p-1)$

$$\begin{aligned} \frac{dw_l}{dt} &\geq \left(-C_3 l^{N-2-\sigma-\{N(\sigma+m)+\theta(m+\sigma)\}/p} + C w_l^{p-(\sigma+m)} \right) \\ &\quad \times \left(\int_{\Omega} |x|^{\theta} u^p \phi_l dx \right)^{(\sigma+m)/p}. \end{aligned}$$

Similar to the arguments of (i) one can prove that w_l , and consequently u , blows up in finite time.

Remark 2.3. The reason for using $\Omega = R^N \setminus B_1$ rather than R^N itself is that if $\theta > 0$, then $\int_{B_1} |x|^{-\theta/(p-1)} dx$ may not converge.

4. PROOF OF THEOREM 1

(i) If $p \leq p_c = \sigma + m + (\sigma + m - 1) + [(\sigma + 2)(1 + s) + \theta]/N$, using the methods similar to those of the last section and the papers [19, 21], it can be proved that every non-trivial solution of (1.1) blows up in finite time. We omit the details.

(ii) If $p > p_c = \sigma + m + (\sigma + m - 1)s + [(\sigma + 2)(1 + s) + \theta]/N$, we shall prove that (1.1) has global positive solutions for the small initial data. By the comparison principle, it is enough to prove this conclusion for the problem (since $s \geq 0$)

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{\sigma} \nabla u^m) + (1+t)^s |x|^{\theta} u^p, & x \in R^N, & t > 0, \\ u(x, 0) &= u_0(x) \geq 0, & x \in R^N, & \end{aligned} \quad (4.1)$$

where the constants m, σ, s, θ, p are as in problem (1.1). We shall deal with the global solutions of (4.1) by using the similarity solutions which take the form

$$u(x, t) = (1+t)^{-\alpha} w(r) \quad \text{with } r = |x|(1+t)^{-\beta},$$

where $\alpha = \{1 + s + \frac{\theta}{\sigma+2}\}/\{p - 1 - \frac{1-\sigma-m}{\sigma+2}\theta\}$, $\beta = \{(1 - \sigma - m)(1 + s) + p - 1\}/\{(p - 1 - \frac{1-\sigma-m}{\sigma+2}\theta)(\sigma + 2)\}$, and w satisfies the following ODE:

$$\begin{aligned} & m(\sigma+1)|w'|^\sigma w'' w^{m-1} + m(m-1)w^{m-2}|w'|^{\sigma+2} \\ & + m \frac{N-1}{r} |w'|^\sigma w' w^{m-1} + \alpha w + \beta r w' + r^\theta w^p = 0, \quad r > 0, \\ & w(0) = \eta > 0, \quad |w'|^\sigma w'(0) = -\lim_{r \rightarrow 0^+} \{r^{\theta+1} w^{p+1-m}(r)/[(N-1)m]\}. \end{aligned} \quad (4.2)$$

We call $w(r)$ a solution of (4.2) in $(0, R(\eta))$ for some $R(\eta) > 0$ if $w(r) > 0$ in $(0, R(\eta))$, $w \in C^2(0, R(\eta))$, and w satisfies the initial condition of (4.2). Under our assumptions it follows that $p > 1 + (1 - \sigma - m)\theta/(\sigma + 2)$, $\alpha > 0$, $\beta > 0$. We observe that a function $\bar{u}(x, t) = (1 + t)^{-\alpha} v(|x|(1 + t)^{-\beta})$ is an upper solution of the equation (4.1) if and only if $v(r)$ satisfies the following inequality:

$$\begin{aligned} & m(\sigma+1)|v'|^\sigma v'' v^{m-1} + m(m-1)v^{m-2}|v'|^{\sigma+2} \\ & + m \frac{N-1}{r} |v'|^\sigma v' v^{m-1} + \alpha v + \beta r v' + r^\theta v^p \leq 0, \quad r > 0. \end{aligned} \quad (4.3)$$

(1) We first discuss the case $\theta \geq 0$. In this case, we try to find an upper solution of (4.1), i.e., the solution of (4.3).

When $\sigma + m < 1$, let $v(r) = \varepsilon(1 + br^k)^{-q}$, where $k = (\sigma + 2)/(\sigma + 1)$, $q = (\sigma + 1)/(1 - \sigma - m)$, and ε and b are positive constants to be determined later. By direct computation we have

$$\begin{aligned} v' &= -\varepsilon q b k r^{k-1} (1 + br^k)^{-q-1}, \\ v'' &= \varepsilon q(q+1)b^2 k^2 r^{2k-2} (1 + br^k)^{-q-2} - \varepsilon q b k(k-1)r^{k-2} (1 + br^k)^{-q-1}. \end{aligned}$$

$v(r)$ satisfies (4.3) if and only if

$$\begin{aligned} & \varepsilon(1 + br^k)^{-q} [\alpha - mN\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1}] + \varepsilon q b k r^k (1 + br^k)^{-q-1} \\ & \times [m\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1} - \beta] + \varepsilon^p r^\theta (1 + br^k)^{-qp} \leq 0. \end{aligned} \quad (4.4)$$

Under our assumptions it follows that $\theta + q(1 - p)k = \theta + (1 - p)(\sigma + 2)/(1 - \sigma - m) < 0$. There exists $a > 0$, such that

$$r^\theta (1 + br^k)^{q(1-p)} \leq a \quad \text{for all } r \geq 0, \quad \text{since } \theta \geq 0. \quad (4.5)$$

Choose $b = b(\varepsilon)$ such that

$$\beta = m\varepsilon^{\sigma+m-1}(bqk)^{\sigma+1};$$

i.e.,

$$b = (qk)^{-1} (\beta m^{-1} \varepsilon^{1-\sigma-m})^{1/(\sigma+1)}.$$

For this choice of b , (4.4) is equivalent to

$$\alpha - N\beta + r^\theta \varepsilon^{p-1} (1 + br^k)^{q(1-p)} \leq 0. \quad (4.6)$$

By (4.5) we see that (4.6) is true if the following inequality holds:

$$\alpha - N\beta + a\varepsilon^{p-1} \leq 0. \quad (4.7)$$

In view of $p > p_c = \sigma + m + (\sigma + m - 1)s + [(\sigma + 2)(1 + s) + \theta]/N$, it follows that $\alpha < N\beta$. Hence, there exists $\varepsilon_0 > 0$ such that (4.7) holds for all $0 < \varepsilon \leq \varepsilon_0$. These arguments show that $v(r) = \varepsilon(1 + br^k)^{-q}$ satisfies (4.3) for all $0 < \varepsilon \leq \varepsilon_0$. Using the comparison principle we get that the solution $u(x, t)$ of (4.1) exists globally provided that $u(x, 0) \leq v(|x|) = \varepsilon(1 + b|x|^k)^{-q}$. And hence, so does the solution of (1.1).

When $\sigma + m = 1$, let $v(r) = \varepsilon \exp\{-br^k\}$, where $k = (\sigma + 2)/(\sigma + 1)$, and ε and b are positive constants to be determined later. By direct computation we know that $v(r)$ satisfies (4.3) if and only if

$$\begin{aligned} & \varepsilon[\alpha - mN(bk)^{\sigma+1}]e^{-br^k} \\ & + \varepsilon bk[m(bk)^{\sigma+1} - \beta]r^k e^{-br^k} + \varepsilon^p r^\theta e^{-pbr^k} \leq 0. \end{aligned} \quad (4.8)$$

Since $\theta \geq 0$, there exists $a > 0$ such that

$$r^\theta \exp\{-(p-1)br^k\} \leq a \quad \text{for all } r \geq 0.$$

Choose b such that $\beta = m(bk)^{\sigma+1}$. Then (4.8) holds provided that

$$\alpha - N\beta + a\varepsilon^{p-1} \leq 0.$$

Similar to the case $\sigma + m < 1$, we have that the solution $u(x, t)$ of (4.1) exists globally provided that $\varepsilon \ll 1$ and $u(x, 0) \leq v(|x|) = \varepsilon \exp\{-b|x|^k\}$. And hence, so does the solution of (1.1).

(2) Next we consider the case $\theta < 0$. If $m = 1$, this problem was discussed by [19] for $\sigma = 0$, and by [21] for $\sigma < 0$. In the following we always assume that $m \neq 1$. Our main purpose is to prove that (4.2) has ground state for the small $\eta > 0$. By the standard arguments one can prove that for any given $\eta > 0$, there exists a unique solution w of (4.2), which is twice continuously differentiable in where $w'(r) \neq 0$.

Denote $R(\eta) = \max\{R | w(r) > 0 \ \forall \ r \in [0, R]\}$. So $0 < R(\eta) \leq +\infty$, and $w(R(\eta)) = 0$ when $R(\eta) < \infty$.

We divide the proof into several lemmas.

LEMMA 2. *The solution $w(r)$ of (4.2) satisfies $w'(r) < 0$ in $(0, R(\eta))$. In addition, if $R(\eta) = +\infty$ then $w(r) \rightarrow 0$ as $r \rightarrow +\infty$.*

Proof. We first prove that $w'(r) < 0$ for $0 < r < R(\eta)$ when $\theta + 1 \leq 0$. Since $|w'|^\sigma w'(0) = -\lim_{r \rightarrow 0^+} \{r^{\theta+1} w^{p+1-m}(r)/[(N-1)m]\} < 0$, one has $w'(r) < 0$ for $r \ll 1$. If there exists $r_0 : 0 < r_0 < R(\eta)$ such that $w'(r) < 0$ in $(0, r_0)$ and $w'(r_0) = 0$, then $(|w'|^\sigma w')' w^{m-1}(r_0) \geq 0$. But by the equation (4.2) we see that

$$m(|w'|^\sigma w')' w^{m-1}(r_0) = -(\alpha w(r_0) + r_0^\theta w^p(r_0)) < 0,$$

a contradiction. When $\theta + 1 > 0$, it follows that $w'(0) = 0$. Using the equation (4.2) one has

$$mN(|w'|^\sigma w')'|_{r=0} = -(\alpha w^{2-m}(0) + \lim_{r \rightarrow 0^+} r^\theta w^{p+1-m}(r)) < 0.$$

Hence $|w'|^\sigma w'(r) < 0$, and consequently $w'(r) < 0$, for all $r \ll 1$. Similar to the case of $\theta + 1 \leq 0$ it follows that $w'(r) < 0$ for all $0 < r < R(\eta)$. If $R(\eta) = +\infty$, since $w'(r) < 0$ and $w(r) > 0$ in $(0, +\infty)$, one has $\lim_{r \rightarrow +\infty} w(r) = L$. If $L > 0$, an integration of (4.2) gives

$$r^{N-1}(m|w'|^\sigma w' w^{m-1} + r\beta w) = -\int_0^r \{\alpha - N\beta + s^\theta w^{p-1}(s)\} s^{N-1} w(s) ds,$$

$$\lim_{r \rightarrow +\infty} \frac{m|w'|^\sigma w' w^{m-1}}{r} = -\frac{\alpha}{N}L - \frac{A}{N},$$

where

$$A = \begin{cases} L^p & \text{if } \theta = 0, \\ 0 & \text{if } \theta < 0, \\ +\infty & \text{if } \theta > 0. \end{cases}$$

It follows that $\lim_{r \rightarrow +\infty} w'(r) = -\infty$, a contradiction. Thus $w(r) \rightarrow 0$ as $r \rightarrow +\infty$. Q.E.D.

LEMMA 3. Let $w(r)$ be the solution of (4.2). Then for any given small $\eta > 0$ there exists $R_0(\eta) > 0$ which satisfies $\lim_{\eta \rightarrow 0^+} R_0(\eta) = +\infty$ and such that

$$w(r) > 0, \quad m|w'|^\sigma w'(r) w^{m-1} + \beta r w(r) > 0, \quad r \in (1, R_0(\eta)). \quad (4.9)$$

Proof. Let $z = \eta - w$; then $z'(r) = -w'(r) > 0$, $0 < z(r) < \eta$, and $z(r)$ satisfies

$$m(\sigma+1)(z')^\sigma z''(\eta-z)^{m-1} - m(m-1)(\eta-z)^{m-2}(z')^{\sigma+2} \\ + m \frac{N-1}{r} (z')^{\sigma+1} (\eta-z)^{m-1} = \alpha(\eta-z) - \beta r z' + r^\theta (\eta-z)^p, \quad r > 0, \\ z(0)=0, \quad (z')^\sigma z'(0) = \lim_{r \rightarrow 0^+} \{r^{\theta+1} (\eta-z)^{p+1-m}(r)/[(N-1)m]\}. \quad (4.10)$$

Since $p > p_c$, one has $N\beta > \alpha$. An integration of (4.10) gives

$$\begin{aligned} & mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} + \beta r^N z \\ &= \int_0^r [(N\beta - \alpha)s^{N-1}z + \alpha\eta s^{N-1} + s^{N+\theta-1}(\eta - z)^p] ds \\ &\leq \frac{\alpha\eta}{N}r^N + (\beta - \frac{\alpha}{N})r^N z(r) + \frac{1}{N+\theta}\eta^p r^{N+\theta}. \end{aligned} \quad (4.11)$$

Since $m \neq 1$ and $-1 < \sigma \leq 0$, we know that if $\sigma + m = 1$ then $\sigma < 0$ and $1 < m < 2$. Denote $R_0(\eta) = \min\{R \mid z(R) = \eta - \eta^a\}$, where $a = \frac{1}{2} \min\{1 - \frac{\sigma}{m-1}, p+1\}$ if $\sigma + m < 1$ and $m > 1$, $a = (p+1)/2$ if $\sigma + m < 1$ and $m < 1$, and $a = \frac{1}{2} \min\{\frac{p+2m-3}{m-1}, p+1\}$ if $\sigma + m = 1$. Then $R_0(\eta) > 0$ and $z(r) \leq \eta - \eta^a < \eta$ for all $0 < r \leq R_0(\eta)$.

We first consider the case $\sigma + m < 1$. From (4.11) it follows that for $0 < r \leq R_0(\eta)$

$$\begin{aligned} mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} &< \frac{\alpha\eta}{N}r^N + (\beta - \frac{\alpha}{N})\eta r^N + \frac{1}{N+\theta}\eta^p r^{N+\theta} \\ &= \beta\eta r^N + \frac{1}{N+\theta}\eta^p r^{N+\theta}. \end{aligned}$$

Denote $b = a$ when $m > 1$, and $b = 1$ when $m < 1$. Using $\eta^a \leq \eta - z \leq \eta$ one has that

$$r^{N-1}(z')^{\sigma+1} < \frac{1}{m} \left\{ \beta\eta^{1+(1-m)b}r + \eta^{p+(1-m)b} \frac{1}{N+\theta} r^{\theta+1} \right\}.$$

Since $\sigma + 1 \leq 1$, it follows that

$$\begin{aligned} z'(r) &< \left\{ \frac{\beta}{m} \eta^{1+(1-m)b}r + \frac{1}{m(N+\theta)} \eta^{p+(1-m)b} r^{\theta+1} \right\}^{1/(\sigma+1)} \\ &\leq C_1 \left\{ (\eta^{1+(1-m)b}r)^{1/(\sigma+1)} + (\eta^{p+(1-m)b} r^{1+\theta})^{1/(\sigma+1)} \right\}. \end{aligned}$$

Integrating this inequality from 0 to $R_0(\eta)$ we have

$$\begin{aligned} \eta &\leq \eta^a + C_2 \left\{ \eta^{(1+(1-m)b)/(\sigma+1)} (R_0(\eta))^{(\sigma+2)/(\sigma+1)} \right. \\ &\quad \left. + \eta^{(p+(1-m)b)/(\sigma+1)} (R_0(\eta))^{(\sigma+\theta+2)/(\sigma+1)} \right\}. \end{aligned}$$

In view of $a > 1$ and $[p + (1-m)b]/(\sigma+1) > [1 + (1-m)b]/(\sigma+1) > 1$, it follows that $R_0(\eta) \rightarrow +\infty$ as $\eta \rightarrow 0^+$.

Using $w(R_0(\eta)) = \eta^a$ and $w(r) \geq \eta^a$ for all $0 \leq r \leq R_0(\eta)$, an integration of (4.2) gives, for $0 \leq r < R_0(\eta)$,

$$\begin{aligned} & mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) \\ &= \int_0^r (N\beta - \alpha) s^{N-1} w(s) ds - \int_0^r s^{N+\theta-1} w^p(s) ds \\ &\geq (N\beta - \alpha) w(R_0(\eta)) \int_0^r s^{N-1} ds - \eta^p \int_0^r s^{N+\theta-1} ds \\ &= \eta^a r^N \left(\beta - \frac{\alpha}{N} - \frac{1}{N+\theta} \eta^{p-a} r^\theta \right). \end{aligned}$$

Since $\theta < 0$, $N\beta > \alpha$, and $p > a$, it follows that

$$mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) > 0, \quad \forall r \in (1, R_0(\eta)). \quad (4.12)$$

Second, we consider the case $\sigma + m = 1$. From (4.11) it follows that, for $0 < r \leq R_0(\eta)$,

$$mr^{N-1}(z')^{\sigma+1}(\eta - z)^{m-1} < \frac{\alpha}{N} r^N (\eta - z) + \frac{1}{N+\theta} \eta^p r^{N+\theta}.$$

Using $\sigma + 1 = 2 - m$ and $1 < m < 2$ we have that

$$z'(r) \leq C \{ (\eta - z) r^{1/(\sigma+1)} + \eta^{(p+(1-m)a)/(\sigma+1)} r^{(1+\theta)/(\sigma+1)} \}. \quad (4.13)$$

Denote $\gamma = [p + (1 - m)a]/(\sigma + 1)$. Integrating (4.13) from 0 to $R_0(\eta)$ we have

$$\begin{aligned} \eta - \eta^a &\leq C \left\{ \eta^a \frac{\sigma+1}{\sigma+2} (R_0(\eta))^{(\sigma+2)/(\sigma+1)} + \eta^\gamma \frac{\sigma+1}{\sigma+2+\theta} (R_0(\eta))^{(\sigma+2+\theta)/(\sigma+1)} \right. \\ &\quad \left. + \frac{\sigma+1}{\sigma+2} \int_0^{R_0(\eta)} r^{(\sigma+2)/(\sigma+1)} z' \right\}. \end{aligned} \quad (4.14)$$

Substituting (4.13) into (4.14) and using the inductive method we have that

$$\begin{aligned} \eta - \eta^a &\leq \eta^a \sum_{n=1}^{+\infty} \frac{1}{n!} A^n + C(\sigma+1)(R_0(\eta))^{(\sigma+2+\theta)/(\sigma+1)} \eta^\gamma \\ &\quad \times \sum_{n=0}^{+\infty} \frac{1}{((n+1)(\sigma+2)+\theta)n!} A^n, \end{aligned} \quad (4.15)$$

where $A = C \frac{\sigma+1}{\sigma+2} (R_0(\eta))^{(\sigma+2)/(\sigma+1)}$. In view of $a > 1$ and $\gamma = [p + (1 - m)a]/(\sigma + 1) > 1$, it follows from (4.15) that $R_0(\eta) \rightarrow +\infty$ as $\eta \rightarrow 0^+$. Similar to the case $\sigma + m < 1$, we have that (4.12) holds. The proof of Lemma 2 is completed. Q.E.D.

Now we prove that, for the case $\theta < 0$, (4.2) has ground state for small η . Choose $\eta_0: \eta_0^{p-1} < N\beta - \alpha$ such that (4.9) holds for all $0 < \eta \leq \eta_0$. Since $p > p_c$, which implies $N\beta > \alpha$, using $\theta < 0$, $R_0(\eta) > 1$, $w(s) < \eta$ and integrating (4.2) from $R_0(\eta)$ to $r(R_0(\eta) < r < R(\eta))$ we have

$$\begin{aligned} & mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r) \\ &= (mr^{N-1}|w'|^\sigma w' w^{m-1} + \beta r^N w(r))|_{r=R_0(\eta)} \\ &\quad + (N\beta - \alpha) \int_{R_0(\eta)}^r s^{N-1} w(s) [N\beta - \alpha - s^\theta w^{p-1}(s)] ds \\ &\geq \int_{R_0(\eta)}^r s^{N-1} w(s) [N\beta - \alpha - \eta^{p-1}] ds \geq 0. \end{aligned} \quad (4.16)$$

In view of $w(r) > 0$ and $w'(r) < 0$ for $0 < r < R(\eta)$, it follows that $R(\eta) = +\infty$ by (4.16). Therefore (4.2) has a ground state.

ACKNOWLEDGMENTS

The authors thank Professor H. A. Levine for directing their attention to [2] and the referees for their helpful comments and suggestions.

REFERENCES

1. D. Andreucci, New results on the Cauchy problem for parabolic systems and equations with strongly nonlinear sources, *Manuscript Math.* **77** (1992), 127–159.
2. D. Andreucci and A. F. Tedeev, A Fujita type results for a degenerate Neumann problem in domains with noncompact boundary, *J. Math. Anal. Appl.* **231** (1999), 543–567.
3. D. G. Aronson and Ph. Benilan, Regularite des solutions de l'equation des milieux poreux dans \mathbf{R}^N , *C. R. Acad. Sci. Paris* **288** (1979), 103–105.
4. C. Bandle and H. A. Levine, On the existence and nonexistence of global solutions of reaction–diffusion equations in sectorial domains, *Trans. Amer. Math. Soc.* **655** (1989), 595–624.
5. K. Deng and H. A. Levine, The role of critical exponents in blow-up theorems: The sequel, *J. Math. Anal. Appl.* **243** (2000), 85–126.
6. E. DiBenedetto, “Degenerate Parabolic Equations,” Springer-Verlag, New York, 1993.
7. M. Escobedo and M. A. Herrero, Boundedness and blow up for a semilinear reaction–diffusion system, *J. Differential Equations* **89** (1989), 176–202.
8. H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124.
9. V. A. Galaktionov and H. A. Levine, A general approach to critical Fujita exponents in nonlinear parabolic problems, *Nonlinear Anal.* **34**, No. 7 (1998), 1005–1027.
10. K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic equations, *Proc. Japan Acad.* **49** (1973), 503–525.
11. A. S. Kalashnikov, Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations, *Uspekhi Mat. Nauk* **42**, No. 2 (1987), 135–176 (in Russian). English translation: *Russian Math. Surveys* **42**, No. 2 (1987), 169–222.

12. K. Kobayashi, T. Siano, and H. Tanaka, On the blowing up problem for semilinear heat equations, *J. Math. Soc. Japan* **29** (1977), 407–424.
13. H. A. Levine, The role of critical exponents in blowup theorems, *SIAM Rev.* **32** (1990), 262–288.
14. H. A. Levine, A Fujita type global existence-global nonexistence theorem for a weakly coupled system of reaction–diffusion equations, *Z. Angew. Math. Phys.* **42** (1990), 408–430.
15. H. A. Levine and P. Meier, A blow up result for the critical exponent in cones, *Israel J. Math.* **67** (1989), 1–8.
16. H. A. Levine and P. Meier, The value of the critical exponent for reaction–diffusion equations in cones, *Arch. Rational Mech. Anal.* **109** (1989), 73–80.
17. Y. W. Qi, On the equation $u_t = \Delta u^\alpha + u^\beta$, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 373–390.
18. Y. W. Qi, Critical exponents of degenerate parabolic equations, *Sci. China Ser. A* **38**, No. 10 (1995), 1153–1162.
19. Y. W. Qi, The critical exponents of parabolic equations and blow-up in R^N , *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 123–136.
20. Y. W. Qi and H. A. Levine, The critical exponent of degenerate parabolic systems, *Z. Angew. Math. Phys.* **44** (1993), 249–265.
21. Y. W. Qi and M. X. Wang, Critical exponents of quasilinear parabolic equations, preprint.
22. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, “Blow-up in Quasilinear Parabolic Equations,” Nauka, Moscow, 1987 (in Russian); English translation: de Gruyter, Berlin, 1995.
23. P. Souplet, Finite time blow-up for a nonlinear parabolic equation with a gradient term and applications, *Math. Methods Appl. Sci.* **19** (1996), 1317–1333.
24. F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38** (1981), 29–40.